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Best Approximation in $(L_1 \oplus R)_\infty$

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We prove necessary and sufficient conditions for X to be a Chebyshev subspace of $(L_1 \oplus R)_\infty$. Moreover, we find a nontrivial Chebyshev subspace of $(L_1 \oplus c_0)_\infty$ when the scalar field is that of the complex numbers.

1. INTRODUCTION

Let F be any Banach space and E a subspace of F. An element e of E is called a best approximation of f in E if and only if it satisfies

$$\|e - f\| = \inf_{e' \in E} \|e' - f\|.$$

E is an *existence subspace* of F if for every f in F there exists at least one best approximation of f in E. E is a *uniqueness subspace* of F if for every f there exists at most one best approximation e of f in E. E is called a *Chebyshev subspace* if E is both an existence and a uniqueness subspace. If E is a Chebyshev subspace, then we can define a function P from F into E such that $P(f)$ is the best approximation of f in E. P is called the *metric projection.* It is known that there exists a nonseparable Banach space which has no proper Chebyshev subspace. But whether there exists a separable Banach space with no proper Chebyshev subspace is still an open question (see [8, p. 31]). It has been conjectured that $(L_1 \oplus R)_\infty$ with the norm $\|(f, \alpha)\| = \max\{\|f\|_1, |\alpha|\}$ has no proper Chebyshev subspace, where L_1 is the space of all real integrable functions on $[0, 1]$. In this article, we find necessary and sufficient conditions for $(L_1 \oplus R)_\infty$ to have a Chebyshev subspace. Namely, $(L_1 \oplus R)_\infty$ has a nontrivial Chebyshev subspace if and only if L_1 has two Chebyshev subspaces Y and Z such that Y is a hyperplane of Z. We do not know whether the real L_1 has these properties. Also, a similar result is true if the real $(L_1 \oplus R)_\infty$ is replaced by the complex $(L_1 \oplus C)_\infty$. It is known that H_1, the Hardy space, and H_1^0, the space of

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functions in H_1 with mean zero, are Chebyshev in L_1 [2]. Hence, $(L_1 \oplus C)_\infty$ has a nontrivial Chebyshev subspace.

It is also well known that L_1 has no finite dimensional or finite codimensional Chebyshev subspace, and c_0, all scalar sequences tending to 0, has no infinite dimensional Chebyshev subspace. We consider the combination $(L_1 \oplus c_0)_\infty$ of L_1 and c_0. We find that $(L_1 \oplus c_0)_\infty$ still has a Chebyshev subspace if L_1 has two Chebyshev subspaces Y and Z such that Y is a hyperplane of Z. Hence, the complex $(L_1 \oplus c_0)_\infty$ has a Chebyshev subspace.

2. Basic Lemmas

Let K denote either R or C. First we recall some elementary and well-known facts which we use in the sequel.

Fact 1. Let F be a Banach space and $\|\cdot\|$ be the norm of F. Then $\|\cdot\|$ is a convex function. Furthermore,

(i) If f, g and h in F and $0 < \gamma < 1$ such that $g = \gamma f + (1-\gamma)h$ then $\|g\| \leqslant \gamma \|f\| + (1-\gamma)\|h\| \leqslant \max(\|f\|, \|h\|)$.

(ii) If f, g and h satisfy the above condition and $\|g\| > \|f\|$, then $\|h\| > \|g\| > \|f\|$.

(iii) If $\|g\| < \|g + cf\|$ for some $c > 0$, then $\|g + c'f\| > \|g + cf\| > \|g\|$ for all $c' > c$.

Fact 2. Let E be subspace of E and e in E. Then e is a best approximation in E of f if and only if 0 is a best approximation of $c(f-e)$ for any $c \neq 0$.

Let $(E \oplus K)_\infty$ be the set $\{(f, \lambda) \mid f \in E \text{ and } \lambda \in K\}$ with the norm $\|(f, \lambda)\| = \max(\|f\|, |\lambda|)$. Here K denotes either R or C. If X is a subspace of $(E \oplus K)_\infty$, then Y and Z are defined as

$$Y = \{f \mid (f, 0) \in X\}$$

and

$$Z = \{f \mid \exists \lambda \in K \text{ such that } (f, \lambda) \in X\}.$$

P is the metric projection from $(E \oplus K)_\infty$ into X if X is Chebyshev, and P' is the metric projection from E into Y when Y is Chebyshev.

Lemma 1. *If X is a nontrivial Chebyshev subspace of $(E \oplus K)_\infty$ then*

(i) $(0, 1)$ *is not in X.*

(ii) *Y is different from Z.*

(iii) *The best approximation of* $(0, 1)$ *in X is not of the form* $(f, 0)$.

Proof. Suppose that $(0, 1)$ is in X. Because X is a proper subspace of $(E \oplus K)_\infty$, there exists a non-zero element $(f, \lambda) \notin X$. By Fact 2, we can suppose that its best approximation is $(0, 0)$. Since

$$\|(f, \lambda) - (0, \lambda)\| = \|(f, 0)\| = \|f\| \leqslant \|(f, \lambda)\|$$

and $(0, 0)$ is the best approximation of (f, λ), $(0, \lambda) = (0, 0)$. Hence, $\lambda = 0$. On the other hand,

$$\|(f, 0) - (0, \|f\|)\| = \|(f, -\|f\|)\| = \|f\|,$$

so $(0, \|f\|)$ is a best approximation of $(f, 0)$. Hence, $\|f\| = 0$. But this contradicts our assumption that $(f, \lambda) \neq (0, 0)$. Therefore, $(0, 1)$ is not in X. This proves (i).

Suppose that $Y = Z$. Then $(0, 0)$ is a best approximation of $(0, 1)$ in X because $X = \{(f, 0) \mid f \in Y\}$ and $\|(0, 1) - (f, 0)\| = \|(-f, 1)\| \geqslant 1$. Since X is nontrivial, there exists $f \neq 0$ such that $(f, 0) \in X$. For $0 < c < 1/\|f\|$,

$$\|(0, 1) - c(f, 0)\| = \|(-cf, 1)\| = 1.$$

Hence, $(cf, 0)$ is another best approximation. This contradicts the fact that X is a Chebyshev subspace. Therefore, Y is different from Z.

By (ii), $Z \neq Y$, so there exists g in X such that $(g, 1)$ is in X. If $0 < c < \min(1/\|g\|, 1)$, then $\|(0, 1) - c(g, 1)\| < 1$. But

$$\|(0, 1) - (f, 0)\| = \|(-f, 1)\| \geqslant 1.$$

Hence, the best approximation of $(0, 1)$ cannot be of the form $(f, 0)$. ∎

Remark 1. If X is Chebyshev, then Y is a hyperplane of Z.

LEMMA 2. *Suppose that X is a nontrivial Chebyshev subspace of* $(E \oplus K)_\infty$. $(0, 0)$ *is the best approximation of* $(h, \lambda) \in (E \oplus K)_\infty$ *in X if and only if h and λ satisfy the following conditions*:

(i) $\|h\| \geqslant |\lambda|$.

(ii) *If* $\|h\| > |\lambda|$ *then* 0 *is the unique best approximation of h in Z* (*so in Y*).

(iii) *If* $\|h\| = |\lambda|$, *then*

(a) 0 *is the unique best approximation of h in Y.*

(b) *If* $(g, 1)$ *in X and* $\|h + cg\| \leqslant \|h\|$ *for some* $c \neq 0$, *then* $|c + \lambda| > |c|$.

Proof. Suppose that $|\lambda| > \|h\|$. Without loss of generality, we can suppose that $\lambda > 0$. Otherwise, we can consider $(\operatorname{sgn} \bar{\lambda})(h, \lambda)$. By Lemma 1, there exists $(g, 1)$ in X. If $0 < c < (\lambda - \|h\|)/\|g\|$, then $\|h - cg\| \leqslant \|h\| + \|cg\| < \lambda$. Hence, for $0 < c < \min((\lambda - \|h\|)/\|g\|, \lambda)$,

$$\|(h, \lambda) - c(g, 1)\| = \max(\|h - cg\|, |\lambda - c|) < \lambda.$$

This contradicts the fact that $(0, 0)$ is the best approximation of (h, λ). Hence, $\|h\| \geqslant |\lambda|$. This proves (i).

Suppose that $\|h\| > |\lambda|$. For each g' in Z, there is α in K such that (g', α) in X. If $0 < |c| < (\|h\| - |\lambda|)/|\alpha|$, then

$$|\lambda - c\alpha| \leqslant |\lambda| + |c\alpha| \leqslant \|h\|.$$

Therefore, if $0 < |c| < (\|h\| - |\lambda|)/|\alpha|$, then

$$\begin{aligned}\|(h, \lambda) - c(g', \alpha)\| &= \max(\|h - cg'\|, |\lambda - c\alpha|) > \|(h, \lambda)\| \\ &= \|h\|.\end{aligned}$$

But $\|h\| > |\lambda - c\alpha|$, so $\|h - cg'\| > \|h\|$. Hence, 0 is the unique best approximation of h in Z. Conversely, suppose that 0 is the best approximation of h in Z. For $|\lambda| < \|h\|$ and (g, α) in X,

$$\begin{aligned}\|(h, \lambda) - (g, \alpha)\| &= \|(h - g, \lambda - \alpha)\| \\ &\geqslant \|h - g\| \\ &> \|h\| = \|(h, \lambda)\|.\end{aligned}$$

So $(0, 0)$ is the best approximation of (h, λ) in X.

Suppose that $\|h\| = |\lambda|$. Then for g' in Y and $g' \neq 0$,

$$\begin{aligned}\|(h, \lambda)\| &< \|(h, \lambda) - (g', 0)\| \\ &= \max(\|h - g'\|, |\lambda|).\end{aligned}$$

Hence, $\|h\| < \|h - g'\|$, and 0 is the best approximation of h in Y. Now, suppose that $(g, 1)$ is in X and $\|h + cg\| < \|h\|$ for some $c \neq 0$. Then

$$\begin{aligned}\|(h, \lambda) + c(g, 1)\| &= \|(h + cg, \lambda + c)\| \\ &> \|(h, \lambda)\| = \|h\| = |\lambda|.\end{aligned}$$

Since $\|h + cg\| \leqslant \|h\|$, $|\lambda + c| > |\lambda|$. The converse direction is trivial. ∎

Remark 2. If X is a uniqueness subspace, then Z is a uniqueness subspace of E.

Now, we suppose that X is a Chebyshev subspace of $(E \oplus K)_\infty$, and h is a fixed element in E. Let $A_1(\lambda)$ be a function from K into K which satisfies

$$P(h, \lambda) = (f_\lambda, A_1(\lambda)).$$

Define, for λ in K,

$$\delta(\lambda) = \operatorname{dist}((h, \lambda), X).$$

A *ball* about λ_0 with *radius* r is $\{\lambda \in K: |\lambda_0 - \lambda| \leqslant r\}$, that is, a *disc* if $K = C$ and an *interval* if $K = R$. For any λ_0 in K, define its *canonical ball*

$$CB(\lambda_0) = \{\lambda \in K: |\lambda - A_1(\lambda_0)| \leqslant \delta(\lambda_0)\}.$$

Then we have that (i) $\lambda_0 \in CB(\lambda_0)$; (ii) $\delta(\lambda) \leqslant \delta(\lambda_0)$ for all λ in $CB(\lambda_0)$; (iii) λ_0 is in the interior of $CB(\lambda_0)$ if and only if

$$\delta(\lambda_0) = \|f_{\lambda_0} - h\| > |A_1(\lambda_0) - \lambda|.$$

Proof. Part (i) follows the definition of $CB(\lambda_0)$ and (ii) follows from the fact

$$\begin{aligned}\|(h, \lambda) - (f_{\lambda_0}, A_1(\lambda_0))\| &= \max(\|h - f_{\lambda_0}\|, |\lambda - A_1(\lambda_0)|)\\ &= \|h - f_{\lambda_0}\| = \delta(\lambda_0).\end{aligned}$$

Suppose that λ_0 is in the interior of $CB(\lambda_0)$. Then there exist λ_1 and λ_2 in $CB(\lambda_0)$ such that $\lambda_0 = \frac{1}{2}(\lambda_1 + \lambda_2)$.

$$\begin{aligned}\delta(\lambda_0) &\leqslant \|(h, \lambda) - \tfrac{1}{2}[(f_{\lambda_1}, A_1(\lambda_1)) + (f_{\lambda_2}, A_1(\lambda_2))]\|\\ &\leqslant \tfrac{1}{2}\|(h, \lambda_1) - (f_{\lambda_1}, A_1(\lambda_1))\| + \tfrac{1}{2}\|(h, \lambda_2) - (f_{\lambda_2}, A_1(\lambda_2))\|\\ &= \tfrac{1}{2}\delta(\lambda_1) + \tfrac{1}{2}\delta(\lambda_2)\\ &\leqslant \delta(\lambda_0).\end{aligned}$$

Hence, $\delta(\lambda_0) = \delta(\lambda_1) = \delta(\lambda_2)$. Since X is Chebyshev,

$$(f_{\lambda_1}, A_1(\lambda_1)) + (f_{\lambda_2}, A_1(\lambda_2)) = 2(f_{\lambda_0}, A_1(\lambda_0)).$$

We have

$$\begin{aligned}\|(h, \lambda_1) - (f_{\lambda_0}, A_1(\lambda_0))\| &= \max(\|h - f_{\lambda_0}\|, |\lambda_1 - A_1(\lambda_0)|)\\ &= \|h - f_{\lambda_0}\| = \delta(\lambda_0) \qquad \text{since} \quad \lambda_1 \in CB(\lambda_0).\end{aligned}$$

Hence, $(f_{\lambda_0}, A_1(\lambda_0)) = (f_{\lambda_1}, A_1(\lambda_1))$. Similarly, $(f_{\lambda_0}, A_1(\lambda_0)) = (f_{\lambda_2}, A_1(\lambda_2))$. Therefore,

$$\begin{aligned} &|\lambda_0 - A_1(\lambda_0)| \\ &\quad = |\tfrac{1}{2}(\lambda_1 - A_1(\lambda_0)) + \tfrac{1}{2}(\lambda_2 - A_1(\lambda_0))| \qquad \text{since } K \text{ is strictly convex and } \lambda_1 \neq \lambda_2 \\ &\quad < \max(|\lambda_1 - A_1(\lambda_0)|, |\lambda_2 - A_1(\lambda_0)|) \\ &\quad \leqslant \delta(\lambda_1) = \delta(\lambda_0) = \delta(\lambda_2). \quad \blacksquare \end{aligned}$$

LEMMA 3. *A_1 is continuous.*

Proof. Let $M = \{\lambda_0 \in K: \delta(\lambda_0) = \inf \delta(\lambda)\}$. Since δ is a continuous and convex function, M is closed and convex. For any λ_0 in K, define a *sub-level set*

$$L(\lambda_0) = \{\lambda \in K: \delta(\lambda) \leqslant \delta(\lambda_0)\}.$$

CLAIM (1). *If λ_0 is not in M, then $CB(\lambda_0)$ is the only ball of radius $\delta(\lambda_0)$ containing λ_0 and contained in $L(\lambda_0)$.*

Proof of (1). That $CB(\lambda_0) \subseteq L(\lambda_0)$ is clear. Suppose that B is another ball of the same radius containing λ_0 and contained in $L(\lambda_0)$. Then $c \circ (B \cup CB(\lambda_0))$ has λ_0 in its interior. By convexity of δ, δ is constant on some small ball contained in $L(\lambda_0)$ and hence λ_0 is in M. This proves (1).

CLAIM (2). *A_1 is continuous on $K - M$.*

Proof of (2). Suppose not. Then there exist $\lambda_n \to \lambda_0 \notin M$ with $A_1(\lambda_n) \to z_0 \neq A_1(\lambda_0)$. Of course $\delta(\lambda_n) \to \delta(\lambda_0)$. Let B be the open ball of radius $\delta(\lambda_0)$ and center z_0. We claim that B is a subset of $L(\lambda_0)$. Suppose that λ is in B. Let n be large enough so that $|\delta(\lambda_n) - \delta(\lambda_0)| < \frac{1}{3}(\delta(\lambda_0) - |\lambda - z_0|)$, and $|A_1(\lambda_n) - z_0| < \frac{1}{3}(\delta(\lambda_0) - |\lambda - z_0|)$. Then

$$\begin{aligned} |\lambda - A_1(\lambda_n)| &\leqslant |\lambda - z_0| + |A_1(\lambda_n) - z_0| \\ &\leqslant |\lambda - z_0| + \tfrac{1}{3}(\delta(\lambda_0) - |\lambda - z_0|) \\ &\leqslant \delta(\lambda_0) - \tfrac{2}{3}(\delta(\lambda_0) - |\lambda - z_0|) \leqslant \delta(\lambda_n). \end{aligned}$$

Hence, $\lambda \in CB(\lambda_n)$. But $CB(\lambda_n)$ is a subset of $L(\lambda_0)$. So B is a subset of $L(\lambda_0)$. Since $\lim_{n\to\infty} \lambda_n = \lambda$, $\lim_{n\to\infty} A_1(\lambda_n) = z_0$ and $\lim_{n\to\infty} \delta(\lambda_n) = \delta(\lambda_0)$, λ is in the closure of B. By the proof of (1), $z_0 = A_1(\lambda_0)$. We get a contradiction. This proves (2).

CLAIM (3). $M \neq \varnothing$.

Proof of (3). Let $c(g, 1)$ be the best approximation of $(0, 1)$ in X. (Note: By Lemma 1, the best approximation of $(0, 1)$ in X is not of the form $(f, 0)$.) By Lemma 2, 0 is the approximation of g in Y. For any $\lambda > \|h\| + 4\|h\|/\|g\|$ and (g', α) in X,

$$\|(h, \lambda) - (g', \alpha)\|$$
$$= \max(\|h - g'\|, |\lambda - \alpha|)$$
$$\geqslant \max(\|g'\| - \|h\|, |\lambda| - |\alpha|)$$
$$\geqslant \max(|\alpha|\,\|g\| - \|h\|, |\lambda| - |\alpha|)$$ since $(g', \alpha) = \alpha(g, 1) + (f, 0)$ and 0 is the best approximation of g in Y
$$\geqslant \|h\| = \|(h, 0)\|.$$

Since δ is a convex function, it attains minimum at some point inside the circle with center 0 and radius $\|h\| + 4\|h\|/\|g\|$. Hence, M is nonempty.

CLAIM (4). *M is a ball of radius $\delta(\lambda_0)$, where λ_0 is any point of M.*

Proof of (4). Let $\lambda_0 \in M$. We show that $M = CB(\lambda_0)$. Surely, $CB(\lambda_0) \subseteq M$. Suppose that λ_1 is in $M - CB(\lambda_0)$. We may assume that the distance ε from λ_1 to $CB(\lambda_0)$ is less than, say, $(10^{-6})\,\delta(\lambda_0)$. Let λ be any point on the line segment from λ_1 to its nearest point λ_2 in $CB(\lambda_0)$. If the interior of $CB(\lambda)$ intersects the interior of $CB(\lambda_0)$, then $(f_{\lambda_0}, A_1(\lambda_0)) = (f_{\lambda_1}, A_1(\lambda))$ by the proof of property (iii) of $CB(\lambda_0)$. But $|\lambda - A_1(\lambda_0)| > \delta(\lambda_0)$ if $\lambda \notin CB(\lambda_0)$. Hence, the interior of $CB(\lambda)$ intersects the interior of $CB(\lambda_0)$ if and only if $\lambda = \lambda_2$. Let λ be any point between λ_1 and λ_2. Since ε, the distance between λ_1 and λ_2, is less than $(10^{-6})\,\delta(\lambda_0)$ and the interior of $CB(\lambda)$ does not intersect the interior of $CB(\lambda_0)$, λ_1 is in the interior of $CB(\lambda)$. Hence, we have $P(h, \lambda) = P(h, \lambda_1) = (f_{\lambda_1}, A_1(\lambda_1))$. By the proof of the property (iii) of $CB(\lambda_0)$,.we have

$$2P(h, \tfrac{1}{2}(\lambda_1 + \lambda_2)) = P(h, \lambda_1) + P(h, \lambda_2).$$

This implies $P(h, \lambda_1) = P(h, \lambda_2)$. We get a contradiction since λ_1 is not in $CB(\lambda_0)$.

CLAIM (5). *A_1 is continuous.*

Proof of (5). A_1 is constant on M and so only the boundary points of M offer any problem. But let λ_0 be such a boundary point and consider a sequence of points outside M converging to λ_0. Now proceed as in (2).

The proof is complete. ∎

COROLLARY 4. *If X is Chebyshev, then Z is Chebyshev.*

Proof. By Claim (4) and Lemma 2. ∎

LEMMA 5. *Λ_1 has a root. Hence, X is Chebyshev, and so Y is Chebyshev.*

Proof. Choose $a > \|h\|$. Let B denote the all of radius a about 0. If $\lambda \in B$, then $\delta(\lambda) \leqslant a$. Hence, the function $\lambda - \Lambda_1(\lambda)$ maps B into itself. Since Λ_1 is continuous, there exists a fixed point, say, λ_0. Hence, $\Lambda_1(\lambda_0) = 0$.

The proof is complete. ∎

3. MAIN RESULT

In this section, we will prove necessary and sufficient conditions for X to be a Chebyshev subspace of $(E \oplus K)_\infty$.

THEOREM 1. *X is a Chebyshev subspace of $(E \oplus K)_\infty$ if and only if*

(i) *Y is a hyperplane of Z, where Y and Z are defined in Section 2.*

(ii) *Both Y and Z are Chebyshev in E.*

Proof. We proved the necessary conditions in Section 2. Now, suppose that Y and Z are Chebyshev subspace of E and Y is a hyperplane of Z. For any $(h, \lambda) \in (E \oplus K)_\infty - X$, we can define a function Λ_2 from K into K by

$$\Lambda_2(\alpha) = \|(h, \lambda) - \alpha(g, 1) - (P'(h - \alpha g), 0)\|,$$

where $(g, 1)$ is in X and P' is the metric projection from E into Y. For $0 < c < 1$ and $\alpha, \beta \in K$,

$$\begin{aligned}
&\Lambda_2(c\alpha + (1-c)\beta) \\
&\quad = \|(h, \lambda) - [c\alpha + (1-c)\beta](g, 1) - (P'(h - [c\alpha + (1-c)\beta] g), 0)\| \\
&\quad \leqslant \|(h, \lambda) - [c\alpha + (1-c)\beta](g, 1) \\
&\qquad - (cP'(h - \alpha g) + (1-c) P'(h - \beta g), 0)\| \\
&\quad \leqslant c \|(h, \lambda) - \alpha(g, 1) - (P'(h - \alpha g), 0)\| \\
&\qquad + (1-c) \|(h, \lambda) - \beta(g, 1) - (P'(h - \beta g), 0)\| \\
&\quad = c\Lambda_2(g) + (1-c) \Lambda_2(\beta).
\end{aligned}$$

Hence, Λ_2 is a convex function. Let B be a ball about 0 with radius $\|h\| + 3|\lambda|$. If α' is not in B, then

$$\Lambda_2(\alpha') = \|(h, \lambda) - \alpha'(g, 1) - (P'(h - \alpha g'), 0)\|$$
$$\geqslant |\lambda - \alpha'| \geqslant \|h\| + 2|\lambda| \geqslant \Lambda_2(0).$$

Hence, Λ_2 attains minimum at some point α in B. We claim that $\alpha(g, 1) + (P'(h - \alpha g), 0)$ is a best approximation of (h, λ) in X. Every element in X has the form $\beta(g, 1) + (y, 0)$. Hence,

$$\|(h, \lambda) - \beta(g, 1) - (y, 0)\|$$
$$\geqslant \|(h, \lambda) - \beta(g, 1) - (P'(h - \beta g), 0)\|$$
$$\text{since} \quad \|h - \beta g - y\| \geqslant \|h - \beta g - P'(h - \beta g)\|$$
$$= \Lambda_2(\beta) \geqslant \Lambda_2(\alpha)$$
$$= \|(h, \lambda) - \alpha(g, 1) - (P'(h - \beta g), 0)\|.$$

$(\alpha g + P'(h - \alpha g), \alpha)$ is a best approximation of (h, λ). Therefore, X is an existence subspace.

Given (h, λ) in $(E \oplus K)_\infty$, by Fact 2 without loss of generality, we can suppose that $(0, 0)$ is a best approximation of (h, λ). By the proof of Lemma 2, either $\|h\| > |\lambda|$ or $\|h\| = |\lambda|$. If $\|h\| > |\lambda|$, then by the proof of Lemma 2 again, 0 is a best approximation of h in Z. Let (f, α) be a non-zero element in X. Hence $f \neq 0$. Since Z is Chebyshev, $\|(h, \lambda) - (f, \alpha)\| \geqslant \|h - f\| > \|h\|$. $(0, 0)$ is the unique best approximation of (h, λ) in X. On the other hand, if $\|h\| = \lambda$, then without loss of generality, we can suppose that $\lambda > 0$. By the proof of Lemma 2, 0 is a best approximation of h in Y. If (f, α) is another best approximation of (h, λ) in X, then $\alpha \neq 0$ since Y is Chebyshev. Hence, for any $0 \leqslant c \leqslant 1$, $c(f, \alpha)$ is a best approximation of (h, λ).

$$\|(h, \lambda)\| = \|(h - f, \lambda - \alpha)\|$$
$$= \|(h - cf, \lambda - c\alpha)\|$$
$$= \|h - cf\| \qquad \text{by the proof of Lemma 2.}$$

$|\lambda| = \|h\| \geqslant |\lambda - \alpha|$. Since K is strictly convex, $|\lambda| > |\lambda - \frac{1}{2}\alpha|$. Hence, $\|h - \frac{1}{2}f\| = \lambda > |\lambda - \frac{1}{2}\alpha|$. By the above argument, $(0, 0)$ is the unique best approximation of $(h - \frac{1}{2}f, \lambda - \frac{1}{2}\alpha)$. Hence, $(\frac{1}{2}f, \frac{1}{2}\alpha)$ is the unique best approximation of (h, λ). We get a contradiction. Therefore, X must be a uniqueness subspace of $(E \oplus K)_\infty$.

Remark 1. X is a uniqueness subspace if and only if one of the following statements i true.

(i) Y is a hyperplane of E and Z is a uniqueness subspace.

(ii) (0, 1) is in X and Z is *very non-proximinal*; that is, no element f in $E-Z$ has an element of best approximation in Z. In this case, X is very non-proximinal.

Remark 2. If $(L_1 \oplus K)_\infty$ has a nontrivial Chebyshev subspace, then Y cannot be a sublattice of L_1.

Proof. Suppose that Y is a sublattice of L_1. Then there exists a measure μ absolutely continuous with respect to the Lebesgue measure such that $L_1(\Sigma, \mu)$ is isometrically isomorphic to Y. First, we claim that μ has no atom: if it is not true, then there exists a μ-measurable set M such that $\mu(M) > 0$, $\chi_M Y \subseteq Y$ has one dimension. $\chi_M Y$ cannot be Chebyshev in $L_1(M)$ since it is a finite dimensional subspace. Hence, there exists f in $L_1(M)$ such that there are two elements, say, 0 and h, in $\chi_M Y$ which are best approximations to f. Let

$$\begin{aligned} \tilde{f}(x) &= f(x) \qquad \text{if} \quad x \in M \\ &= 0 \qquad \text{if} \quad x \notin M. \end{aligned}$$

Then 0 and h are best approximations to $\tilde{f}$ in Y since

$$\|\tilde{f}(x) - y\| \geqslant \|f(x) - \chi_M y\| \geqslant \|\tilde{f}(x)\|_1.$$

This contradicts the fact that Y is Chebyshev. Therefore, μ has no atom.

Suppose that Z is the subspace generated by Y and g, where g is in $P'^{-1}(0)$. We claim that Z is not Chebyshev. Let $\tilde{M}$ be a μ-measurable set such that $\mu(\tilde{M}) = 0$ and the Lebesgue measure restricted to the complement of $\tilde{M}$ is absolutely continuous with respect to μ. Let M_1 and M_2, two μ-measurable sets, be a partition of $\tilde{M}^c$ such that $\int_{M_1} |g(x)|\, dx = \int_{M_2} |g(x)|\, dx$. This can be done because μ has no atom. Let M_3 and M_4 be a partition of $\tilde{M}$ such that $\int_{M_3} |g(x)|\, dx = \int_{M_4} |g(x)|\, dx$. Because $\chi_{M_1} Y \subseteq Y$, $\chi_{M_2} Y \subseteq Y$ and $\mu(\tilde{M}) = 0$, 0 is a best approximation of $\alpha_1 \chi_{M_1} g + \alpha_2 \chi_{M_2} g + \alpha_3 \chi_{M_3} g + \alpha_4 \chi_{M_4} g$, where $\alpha_i \in K$. Let $\tilde{g}$ be defined by

$$\begin{aligned} \tilde{g}(x) &= g(x) \qquad x \in M_1 \cup M_3 \\ &= -g(x) \qquad x \in M_2 \cup M_4. \end{aligned}$$

Then for $-1 \leqslant \alpha \leqslant 1$

$$\|\tilde{g}-\alpha g\|_1 = \int_0^1 |\tilde{g}(x)-\alpha g(x)|\,dx$$

$$= \int_{M_1\cup M_3} |\tilde{g}(x)-\alpha g(x)|\,dx + \int_{M_2\cup M_4} |\tilde{g}(x)-\alpha g(x)|\,dx$$

$$= \int_{M_1\cup M_3} |g(x)-\alpha g(x)|\,dx + \int_{M_2\cup M_4} |-g(x)-\alpha g(x)|\,dx$$

$$= (1-\alpha)\int_{M_1\cup M_2} |g(x)|\,dx + (1+\alpha)\int_{M_2\cup M_4} |g(x)|\,dx$$

$$= \int |g(x)|\,dx = \int |\tilde{g}(x)|\,dx.$$

Hence, $\|\tilde{g}\|_1 = \|\tilde{g}-g\|_1 \leqslant \|\tilde{g}-\beta g\|_1 \leqslant \|\tilde{g}-\beta g - y\|_1$ for $\beta \in K$ and $y \in Y$. g and 0 are best approximations of $\tilde{g}$ in Z and Z is not Chebyshev.

Remark 3. If μ is a measure without atoms, then $L_1(\Sigma, \mu)$ has no finite codimensional Chebyshev subspace. Hence, if X is Chebyshev, then Z is not a sublattice.

Remark 4. (I am indebted to P. Morris, who showed me a nontrivial Chebyshev subspace of real L_1.) Let $G = \{f \mid f \in L_1$ such that $f(x) = f(x+\frac{1}{3}) = f(x+\frac{2}{3})$ for $0 \leqslant x \leqslant \frac{1}{3}\}$. For any $h \in L_1[0,1]$, the best approximation f on h in G is defined by

$$f(x) = f(x+\tfrac{1}{3}) = f(x+\tfrac{2}{3}) = h(x_2),$$

where x_1, x_2, x_3 are $x, x+\frac{1}{3}, x+\frac{2}{3}$ such that $h(x_1) \leqslant h(x_2) \leqslant h(x_3)$, and $0 \leqslant x \leqslant \frac{1}{3}$. The metric projection is not linear because both $\chi_{[0,1/3)}$ and $\chi_{[1/3,2/3)}$ have 0 as the best approximation, but $\chi_{[0,2/3)}$ has $\chi_{[0,1)}$ as its best approximation. But G is a sublattice; hence, Y cannot be G (respectively, Z cannot be G) when X is Chebyshev in $(L_1 \oplus R)_\infty$.

Remark 5. It is known that the Hardy space H_1 and H_1^0 (all functions in H_1 with mean zero) are Chebyshev in complex L_1 [2]. Hence, $(L_1 \oplus C)_\infty$ has a nontrivial Chebyshev subspace.

COROLLARY 1.1. *Suppose Y is a Chebyshev subspace of E and $f_1, f_2,\dots, f_n \in E - Y$. If for any subset A of $\{f_1, f_2,\dots, f_n\}$, the subspace generated by $Y \cup A$ is Chebyshev, and Y is an n codimensional subspace of the generated by $Y \cup \{f_1, f_2,\dots, f_n\}$, then X, the subspace generated by $\{(f, (0, 0,\dots, 0)) \mid f \in Y\}$ and $\{(f_i, e_i) \mid i = 1, 2,\dots, n\}$, is Chebyshev in $(E \oplus K^n)_\infty$, where e_i is the natural basis of K^n.*

Proof. Since $(E \oplus K^n)_\infty = ((E \oplus K^{n-1})_\infty \oplus K)_\infty$, by induction it is enough to prove that $n = 2$. By assumption, Y is different from the subspace Z_1 generated by Y and f_1, and the subspace Z_2 generated by Y and f_2 is different from the subspace Z_3 generated by Y, f_1, and f_2. By Theorem 1, the subspace Z_4 generated by $\{(y, 0) \mid y \in Y\} \cup \{(f_1, 1)\}$, and the subspace Z_5 generated by $\{(y, 0) \mid y \in Y\} \cup \{(f_1, 1), (f_2, 0)\}$ are Chebyshev in $(E \oplus K)_\infty$. Hence, the subspace generated by $\{(y, 0, 0) \mid y \in Y\} \cup \{(f_1, 1, 0), (f_2, 0, 1)\}$ is Chebyshev in $(E \oplus K^2)_\infty$. ∎

4. An Example of a Chebyshev Subspace of $(L_1 \oplus c_0)_\infty$

Since $L_1([0, \infty))$ is isometrically isomorphic to $L_1([0, 1))$, we can consider $(L_1([0, \infty)) \oplus c_0)_\infty$, where c_0 is all scalar sequences converging to zero, and $(L_1 \oplus c_0)_\infty$ has the norm $\|f \oplus (c_i)\|_\infty = \max(\|f\|_1, \|(c_i)\|_\infty)$. We need the following lemma.

Lemma 6. *If Y_i is a Chebyshev subspace of X_i, then $(\oplus Y_i)_{l_1}$ is a Chebyshev subspace of $(\oplus X_i)_{l_1}$. Indeed, for $(x_i) \in (\oplus X_i)_{l_1}$ if y_i is the approximation of x_i in Y_i, then (y_i) is the best approximation of (x_i) in $(\oplus Y_i)_{l_1}$.*

Proof. Since $\|y_i\| \leqslant 2\|x_i\|$, (y_i) is an element of $(\oplus Y_i)_{l_1}$. Suppose that $(y'_i) \in (\oplus Y_i)_{l_1}$ and $(y'_i) \neq (y_i)$. Then

$$\begin{aligned} \|(x_i - y'_i)\| &= \sum_{i=1}^{\infty} \|x_i - y'_i\| \\ &> \sum_{i=1}^{\infty} \|x_i - y_i\| \\ &= \|(x_i - y_i)\|. \end{aligned}$$

Hence, $(\oplus Y_i)_{l_1}$ is a Chebyshev subspace of $(\oplus X_i)_{l_1}$. ∎

Example. Let X be the space of pairs $f \oplus (\beta_i)$, where the restriction of f to $[n-1, n]$ is in H_1 for each n and where

$$\beta_i = \int_{i-1}^{i} f(x)\,dx \qquad \text{for all } i.$$

Then X is a Chebyshev subspace of $(L_1([0, \infty)) \oplus c_0)_\infty$.

Proof. It is easy to see that X is a closed subspace of $(L_1 \oplus c_0)_\infty$. Let Z be the space

$$\{f \mid f|_{[n-1,n]} \text{ is in } H_1 \text{ for all } n\} \qquad \text{(after shift)}.$$

By Lemma 6, Z is a Chebyshev subspace of L_1. Given $g \oplus (\alpha_i)$ in $(L_1 \oplus c_0)_\infty$, without loss of generality we can suppose that 0 is the best approximation of g in Z. If $g \oplus (\alpha_i) = 0 \oplus 0$, then it is done. Hence, we can suppose that either $g \neq 0$ or there exists n such that $\alpha_n \neq 0$. Let N e large enough such that for any $i > N$, we have $\frac{1}{2}\|g\|_1 \geqslant |\alpha_i|$ if $g \neq 0$ or $\frac{1}{2}|\alpha_n| \geqslant |\alpha_i|$ if $g = 0$. Let Y_N be the space of pairs $f \oplus (\beta_i)$ such that the restriction of f to $[i-1, i]$ is in H_1 if $i \leqslant N$, and the restriction is 0 if $i > N$, and

$$\beta_i = \int_{i-1}^{i} f(x)\,dx \qquad \text{for all } i.$$

By Corollary 1.1 of Theorem 1 and Lemma 6, Y_N is a Chebyshev subspace of $(L_1[0, N) \oplus C^N)_\infty$. Let $f \oplus (\beta_i)$ be the best approximation of $g \oplus (\alpha_1, \alpha_2, \dots, \alpha_N)$ in Y_N.

CLAIM (1). $\|g - f\|_1 > |\alpha_i|$ *for* $i > N$.

Proof of (1). Since 0 is the best approximation g in Z, $\|g - f\|_1 \geqslant \|g\|_1$. Hence, if $\|g\|_1 \neq 0$, then $\|g - f\|_1 \geqslant \|g\|_1 > |\alpha_i|$ for $i > N$. So we can suppose that $g = 0$. Since $\int_{n-1}^{n} f(x)\,dx = \beta_n$, $\|f\| \geqslant |\beta_n|$. On the other hand, since $0 \oplus 0$ is the best approximation of $(g - f, (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \dots, \alpha_N - \beta_N))$, by Lemma 2,

$$\|f\| \geqslant |\alpha_n - \beta_n| \qquad \text{since} \quad g = 0.$$

Hence, $2\|f\| \geqslant |\alpha_n - \beta_n| + |\beta_n| \geqslant |\alpha_n| > 2|\alpha_i|$ for $i > N$.

CLAIM (2). $f \oplus (\beta_1, \beta_2, \dots, \beta_N, 0, 0, \dots)$ *is the best approximation of* $g \oplus (\alpha_i)$.

Proof of (2). Let $h \oplus (\gamma_i)$ be any element in X. Then

$$\begin{aligned} h \oplus (\gamma_i) = {} & h\chi_{[0,N)} \oplus (\gamma_1, \gamma_2, \dots, \gamma_N, 0, 0, \dots) \\ & + h\chi_{[N,\infty)} \oplus (0, 0, \dots, 0, \gamma_{N+1}, \gamma_{N+2}, \dots). \end{aligned}$$

If $h \oplus (\gamma_i) \neq f \oplus (\beta_1, \beta_2, \dots, \beta_N, 0, 0, \dots)$ then either

(1) $\quad h\chi_{[0,N)} \oplus (\gamma_1, \gamma_2, \dots, \gamma_N, 0, 0, \dots) \neq f \oplus (\beta_1, \beta_2, \dots, \beta_N, 0, 0, \dots)$

or

(2) $\quad h\chi_{[N,\infty)} \oplus (0, \dots, 0, \gamma_{N+1}, \gamma_{N+2}, \dots) \neq 0 \oplus (0, 0, \dots)$.

Suppose that (1) is true. Since $f \oplus (\beta_1, \beta_2, \ldots, \beta_N)$ is the best approximation of $g \oplus (\alpha_1, \alpha_2, \ldots, \alpha_N)$ in Y_N, either $\|g-f\|_1 < \|g - h\chi_{[0,N)}\|_1$ or there exists $1 \leqslant m \leqslant N$ such that $\|g-f\|_1 < |\alpha_m - \gamma_m|$. If $\|g-f\|_1 < |\alpha_m - \gamma_m|$, then

$$\begin{aligned}
&\|g \oplus (\alpha_1, \alpha_2, \ldots) - f \oplus (\beta_1, \beta_2, \ldots, \beta_N, 0, 0, \ldots)\| \\
&\quad = \|g-f\|_1 \\
&\quad < |\alpha_m - \gamma_m| \\
&\quad \leqslant \|g \oplus (\alpha_1, \alpha_2, \ldots) - h \oplus (\gamma_1, \gamma_2, \ldots)\|.
\end{aligned}$$

If $\|g-f\|_1 < \|g - h\chi_{[0,N)}\|_1$ then

$$\begin{aligned}
\|g-f\|_1 &< \|g - h\chi_{[0,N)}\|_1 \\
&= \|(g-h)\chi_{[0,N)}\|_1 + \|g\chi_{[N,\infty)}\|_1 \\
&\leqslant \|(g-h)\chi_{[0,N)}\|_1 \\
&\quad + \|g\chi_{[N,\infty)} - h\chi_{[N,\infty)}\|_1 \qquad \text{since by Lemma 6, 0 is the best approximation of } g\chi_{[N,\infty)} \text{ in } Z \\
&= \|g-h\|_1.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|g \oplus (\alpha_1, \alpha_2, \ldots) - h \oplus (\gamma_1, \gamma_2, \ldots)\| \\
&\quad < \|g \oplus (\alpha_1, \alpha_2, \ldots) - f \oplus (\beta_1, \beta_2, \ldots, \beta_N, 0, 0, \ldots)\|.
\end{aligned}$$

If (1) is not true, then (2) must be true. And it implies $h\chi_{[N,\infty)} \neq 0$. Hence,

$$\begin{aligned}
\|g-f\| &= \|g\chi_{[0,N)} - h\chi_{[0,N)}\|_1 + \|g\chi_{[N,\infty)}\|_1 \\
&< \|g\chi_{[0,N)} - h\chi_{[0,N)}\|_1 + \|g\chi_{[N,\infty)} - h\chi_{[N,\infty)}\|_1 \\
&= \|g-h\|_1.
\end{aligned}$$

Hence, $f \oplus (\beta_1, \beta_2, \ldots, \beta_N, 0, 0, \ldots)$ is the unique best approximation of $g \oplus (\alpha_1, \alpha_2, \ldots)$ in X. X is Chebyshev. ∎

Remark 1. If the real L_1 has two Chebyshev subspaces Y and Z such that Y is a hyperplane of Z, then by the above method we can construct a Chebyshev subspace of $(L_1 \oplus c_0)_\infty$.

Since the unit ball of L_1 has no extreme point, L_1 has no coreflexive Chebyshev subspace. But we do not know whether L_1 has a reflexive Chebyshev subspace. It is also conjectured that $(L_1 \oplus L_1 \oplus L_1 \oplus \cdots)_\infty$ has no Chebyshev subspace. We have the following open problem.

PROBLEM. Does $(L_1 \oplus L_1)_\infty$ have a nontrivial Chebyshev subspace? Find necessary and sufficient conditions for X to be a Chebyshev subspace of $(L_1 \oplus L_1)_\infty$.

Remark 2. Suppose that X is a Chebyshev subspace of $(L_1 \oplus L_1)_\infty$. Let

$$Y_1 = \{f \mid (f, 0) \in X\}, \text{ and } Z_1 = \{f \mid \exists g \in L_1 \text{ such that } (f, g) \in X\},$$
$$Y_2 = \{f \mid (0, f) \in X\}, \text{ and } Z_2 = \{f \mid \exists g \in L_1 \text{ such that } (g, f) \in X\}.$$

Then (i) Z_1 and Z_2 are uniqueness subspaces; (ii) If $Y_1 \neq \{0\}$ (respectively $Y_2 \neq \{0\}$), Z_2 (respectively Z_1) is very non-proximinal. (Z_1 or Z_2 may not be closed.) (iii) If $0 \oplus 0$ is the approximation of $f \oplus g$ in X and $\|f\|_1 > \|g\|_1$ (respectively $\|g\|_1 > \|f\|_1$), then 0 is the best approximation of f (respectively, g) in Z_1 (respectively Z_2). If there exists an element $f \oplus g$ with the above property, then $Y_2 = \{0\}$ (respectively $Y_1 = \{0\}$).

Remark 3. $L_2[0, 1]$ has an infinite dimensional subspace E such that for any $f \in E$ we have $\|f\|_2 > 10\|f\|_1$. It is known that $L_2[0, 1]$ with the new norm

$$|||f||| = \max\{\|f\|_1, \tfrac{1}{3}\|f\|_2\}$$

has no finite dimensional Chebyshev subspace. But

$$E^\perp = \{g \mid \langle f, g\rangle = 0 \text{ for all } f \in E\}$$

is a Chebyshev subspace of $(L_2[0, 1], |||\cdot|||)$. Particularly, for each $f \in E$, 0 is the best approximation of f in $E^\perp$.

Proof. Let $f \in E$. Then

$$|||f||| = \max\{\|f\|_1, \tfrac{1}{3}\|f\|_2\} = \tfrac{1}{3}\|f\|_2.$$

For any $g \in E^\perp$,

$$\|f + g\|_2 \geqslant \|f\|_2.$$

So we have $|||f + g||| > |||f|||$, and $E^\perp$ is a Chebyshev subspace of $(L_2[0, 1], |||\cdot|||)$.

We claim that for any separable infinite dimensional Banach space F, $((L_2[0, 1], |||\cdot|||) \oplus F)_\infty$ has a Chebyshev subspace.

Proof. Let $f_1, f_2, \ldots$ be a orthonormal basis of E with the norm $\|\cdot\|_2$ and $g_1, g_2, \ldots$ be linear independent such that $\overline{\text{span}\{g_1, g_2, \ldots\}} = F$. Let X be the subspace generated by $\{(f, 0) \mid f \in E^\perp\} \cup \{(f_i, g_i/n^2\|g_i\|), i = 1, 2, \ldots\}$. Since X is isomorphic to $L_2[0, 1]$, it is an existence subspace.

CLAIM (1). *If* $(0, 0)$ *is a best approximation of* (f, g), *then* $|||f||| \geqslant \|g\|_F$.

Proof of (1). Suppose that $\|g\|_F > |||f|||$. Since $\overline{\text{span}\{g_1, g_2,...\}} = F$, there exist $f' \in L_2[0, 1]$ and $g' \in F$ such that $(f', g') \in X$ and $\|g - g'\|_F < \frac{1}{2}\|g\|_F$. If $0 < c < \min(1, (\|g\|_F - |||f|||)/2\,|||f'|||)$, then

$$\|(f, g) - c(f', g')\| < \|g\|_F.$$

This contradicts the fact that $(0, 0)$ is a best approximation of (f, g) in X. Hence, $|||f||| \geqslant \|g\|_F$.

CLAIM (2). *X is a Chebyshev subspace of* $((L_2[0, 1], |||\cdot|||) \oplus F)_\infty$.

Proof of (2). Suppose not. Let $(0, 0)$ and (f', g') be two best approximations of (f, g) in X. Hence,

$$|||f||| = |||f - f'||| = \|(f, g)\|.$$

Since $\{(f, 0) \mid f \in E^\perp\}$ is contained in X, f and $f - f'$ belong to E and $\|f\|_2 = |||f||| = |||f - f'||| = \|f - f'\|_2$. But $\frac{1}{2}(f', g')$ is another best approximation of (f, g) in X; hence,

$$\begin{aligned} \|f - \tfrac{1}{2}f'\|_2 &= |||f - \tfrac{1}{2}f'||| \\ &= \|f\|_2 \\ &= \|f - f'\|_2. \end{aligned}$$

This contradicts the fact that $(L_2(0, 1), \|\cdot\|_2)$ is strictly convex. So X is Chebyshev. ∎

Hence, $(\oplus\,(L_2[0, 1], |||\cdot|||))_{c_0}$ has a Chebyshev subspace.

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